Second test, answers

Math 120 Calculus I, Clark University

Fall, 2007

1. [12] Let $f(x)$ be a function.

a. True or false?

$f'(1) = \lim_{h\to 1} \frac{f(1+h)-f(1)}{h}$. False. Should be $\lim_{h\to 0}$.

$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$. True.

$f'(0) = \lim_{h\to 0} \frac{f(h)-0}{0}$. False. The expression is nonsense

$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$. True.

b. Use that definition to compute $f'(2)$ if $f(x) = x^2 + 5$.

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{((2+h)^2+5)-(2^2+5)}{h} \\ &= \lim_{h\to 0} \frac{(4+4h+h^2+5)-(4+5)}{h} \\ &= \lim_{h\to 0} \frac{4h+h^2}{h} \\ &= \lim_{h\to 0} (4+h) = 4 \end{aligned}$$

2. [64] Compute the derivatives.

a. $\frac{d}{dx}\left(x^4 - 10x^3 + \frac{x^2}{2} - \frac{2}{x} + \frac{3}{2}\right)$
$= 4x^3 - 30x^3 + x + \frac{2}{x^2}$.

Note that you don't need the quotient rule for this derivative. Treat the term $x^2/2$ as the constant $\frac{1}{2}$ times x^2 so that its derivative is $\frac{1}{2}$ times $2x$, which equals x. Treat the term $\frac{2}{x}$ as $2x^{-1}$ so that its derivative is 2 times $-1x^{-2}$. The final term $\frac{3}{2}$ is a constant, so its derivative is 0.

b. $\frac{d}{dx}(1-3x)^5 = 5(1-3x)^4(-3)$

You need the power rule and the chain rule. The derivative of $(1-3x)^5$ is $5(1-3x)^4$ times the derivative of $1-3x$.

c. $\frac{d}{dx}\left(2\sqrt{x} - \sqrt{2x}\right) = \frac{1}{\sqrt{x}} - \frac{2}{\sqrt{2x}}$

The derivative of the first term, $2\sqrt{x}$ is 2 times the derivative of $\sqrt{x}$, that is, 2 times $\frac{1}{2\sqrt{x}}$. If you prefer fractional exponents, write $\sqrt{x}$ as $x^{1/2}$, and its derivative as $\frac{1}{2}x^{-1/2}$.

If you leave the second term, $\sqrt{2x}$ in the form it's in, or if you change it to $(2x)^{1/2}$, then you'll need the chain rule to find the derivative. The derivative of the inner function, $2x$, is 2.

d. $\frac{d}{dx}\frac{2x+3}{3x+4} = \frac{2(3x+4)-(2x+3)\,3}{(3x+4)^2}$

All that's needed for this one is the quotient rule.

e. $\frac{d}{dx}(\cos 3x \sin 4x - \cos 3 \sin 4)$
$= (-\sin 3x)\,3 \sin 4x + \cos 3x\,(\cos 4x)\,4$

First note that $\cos 3 \sin 4$ is a constant, so its derivative is 0.

The first term, $\cos 3x \sin 4x$, is a product, so the product rule is needed. Remember that the derivative of uv is not $u'v'$. The derivative is $u'v + uv'$. So the derivative will be $(\cos 3x)'(\sin 4x) + (\cos 3x)(\sin 4x)'$. Since both $\cos 3x$ and $\sin 4x$ are compositions of functions, you'll have to use the chain rule when differentiating them.

f. $\frac{d}{dx}(\tan x - \sec^2 x) = \sec^2 x - 2\sec x \sec x \tan x$

Remember that the derivative of $\tan x$ is $\sec^2 x$, and the derivative of $\sec x$ is $\sec x \tan x$.

The difficult part of this one is recognizing that you need the power rule and chain rule when differentiating $\sec^2 x$. Since $\sec^2 x$ is an abbreviation for $(\sec x)^2$, its derivative is $2\sec x$ times the derivative of $\sec x$.

g. $\frac{d}{dx}(\sin^2 x + \cos^2 x) = 0$, since $\sin^2 x + \cos^2 x = 1$.

If you didn't notice that the function being differentiated was the constant 1, you could still take the derivative. Like in part **f**, the power rule and the chain rule are needed to differentiate the square of a trig function. You'll get $2\sin x \cos x + 2\cos x(-\sin x)$ which simplifies to 0.

h. $\frac{d}{dx}\left(1 + \sqrt{1+\sqrt{x}}\right) = \frac{1}{2\sqrt{1+\sqrt{x}}}\,\frac{1}{2\sqrt{x}}$

You'll need the chain rule to differentiate this. You'll find that the derivative of $\sqrt{1+\sqrt{x}}$ is $\frac{1}{2\sqrt{1+\sqrt{x}}}$ times the derivative of $1+\sqrt{x}$.

3. [10] The function $f(x) = 4 - x^2$ is graphed on the test.

a. Find the equation of the tangent line to the graph of $f(x)$ at $x = -1$, and draw the tangent line on the graph.

$f'(x) = -2x$, so $f'(-1) = 2$, so the slope of the tangent line is 2, and it passes through the point $(x, y) = (-1, 3)$. Therefore the equation of the tangent line is $\dfrac{y-3}{x+1} = 2$, or $y = 2x + 5$.

b. Find a value x_0 so that the slope of the tangent line at x_0 is perpendicular to the tangent line found in part **a**. Sketch that tangent line at x_0.

Since the slope in part **a** was 2, a perpendicular line will have slope $-\frac{1}{2}$. We need to find where $f'(x) = -\frac{1}{2}$, that is, we need to solve the equation $-2x = -\frac{1}{2}$. Then $x = \frac{1}{4}$. Therefore, the perpendicular line passes through $(\frac{1}{4}, \frac{63}{16})$ and has slope $-\frac{1}{2}$. It's equation is $\dfrac{y - 63/16}{x - 1/4} = -\dfrac{1}{2}$, or $y = -\frac{1}{2}x + \frac{65}{16}$.

4. [6] **a.** Draw the graph of a function which is defined and differentiable for $-2 \le x \le 2$, satisfies $0 \le f(x) \le 3$, and for which $f'(1) = f'(-1) = 2$.

Just make sure that the graph of your function lies inside the rectangle described by the inequalities and that the tangents of the curve at $x = 1$ and $x = -1$ have slope 2. Note that a straight line with that slope won't work since it will stick out of the rectangle.

b. Draw the graph of a function which is defined and continuous for $-2 \le x \le 2$, and satisfies all of the following:

$f(x)$ is differentiable for all $x \ne 0$.
$f(x)$ is not differentiable for $x = 0$.
$f'(-2) = f'(-1) = f'(1) = f'(2) = 0$.

The graph should be smooth except at $x = 0$ where there should be a corner in the graph. It should have horizontal tangents at $x = -2, -1, 1,$ and 2. One way to do that is to start with a horizontal line and adjust it near $x = 0$ to insert a corner or cusp.

5. [8] Suppose that $f(x)$ and $g(x)$ are differentiable functions and that $f(2) = g(1) = 2$, $f(1) = g(2) = 1$, $f'(2) = g'(2) = 5$, and $f'(1) = g'(1) = 4$.

For each of the following, find the derivatives at $x = 2$.

a. $f(1) + g(2)$. This is a constant. Its derivative everywhere is 0.

b. $(f(x) + 1)(g(x) - 1)$. You'll need the product rule. Its derivative with respect to x is

$$f'(x)(g(x) - 1) + (f(x) + 1)g'(x),$$

so at $x = 2$ the derivative is

$$f'(2)(g(2) - 1) + (f(2) + 1)g'(2)$$

which equals $5(1 - 1) + (2 + 1)5 = 15$.

c. $f(g(x))$. You'll need the chain rule. Its derivative at 2 is $f'(g(2))g'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

d. $g(f(x))$. Again, the chain rule. Its derivative at 2 is $g'(f(2))f'(2) = g'(2) \cdot 5 = 5 \cdot 5 = 25$.