Calculus II – Math 121

Test #1
Spring 2007

Scale: 90–100 A, 80–89 B, 70–79 C, 50–69 D. Median 82.

1. 10 points each] Compute the following definite integrals:

a. $\displaystyle\int_0^1 (9t^3 - 4t^2)\,dt$

Use the power rule $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ to evaluate this.

$$\begin{aligned}\int_0^1 (9t^3 - 4t^2)\,dt &= \left(\frac{9t^4}{4} - \frac{4t^3}{3}\right)\Bigg|_0^1 \\ &= \left(\frac{9}{4} - \frac{4}{3}\right) - (0-0) = \frac{11}{12}\end{aligned}$$

b. $\displaystyle\int_{-\pi/2}^{\pi/2} (\cos x - \sin x)\;dx$

Recall that the integral of $\cos x$ is $\sin x$ while the integral of $\sin x$ is $-\cos x$.

$$\begin{aligned}&\int_{-\pi/2}^{\pi/2} (\cos x - \sin x)\;dx \\ =\;& (\sin x + \cos x)\Big|_{-\pi/2}^{\pi/2} \\ =\;& (\sin \tfrac{\pi}{2} + \cos \tfrac{\pi}{2}) - (-\sin \tfrac{\pi}{2} + \cos -\tfrac{\pi}{2}) \\ =\;& (1+0) - (-1+0) = 2\end{aligned}$$

2. [10 points each] Compute the following indefinite integrals:

a. $\displaystyle\int \left(\frac{t^2 - t^{-2}}{t^4}\right) dt$

There is no quotient rule for integrals, so you have to simplify the integrand, then use the power rule.

$$\begin{aligned}\int \left(\frac{t^2 - t^{-2}}{t^4}\right) dt &= \int (t^{-2} - t^{-6})\,dt \\ &= \frac{t^{-1}}{-1} - \frac{t^{-5}}{-5} + C \\ &= -\frac{1}{t} + \frac{1}{5t^5} + C\end{aligned}$$

Be sure to always include the differential, dt in this problem, until you've taken the integral. Once you've taken the integral, always include the "$+C$" with indefinite integrals.

b. $\displaystyle\int \left(\frac{\sqrt{x}}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{x}}\right) dx$

Again, there is no quotient rule for integrals. Simplify the integrand first.

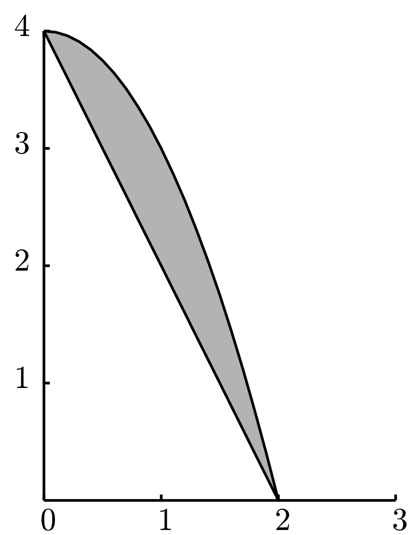
$$\begin{aligned}\int \left(\frac{\sqrt{x}}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{x}}\right) dx &= \int \left(\frac{1}{\sqrt{2}}\,x^{1/2} + \sqrt{2}\,x^{-1/2}\right) dx \\ &= \frac{1}{\sqrt{2}}\,\frac{x^{3/2}}{3/2} + \sqrt{2}\,\frac{x^{1/2}}{1/2} + C \\ &= \frac{\sqrt{2}}{3}\,x^{3/2} + 2\sqrt{2}\,x^{1/2} + C\end{aligned}$$

c. $\displaystyle\int (x^2+4)(x+1)\,dx$

There is no product rule for integrals. Simplify the integrand first.

$$\begin{aligned}\int (x^2+4)(x+1)\,dx &= \int (x^3 + x^2 + 4x + 4)\,dx \\ &= \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3 + 2x^2 + 4x + C\end{aligned}$$

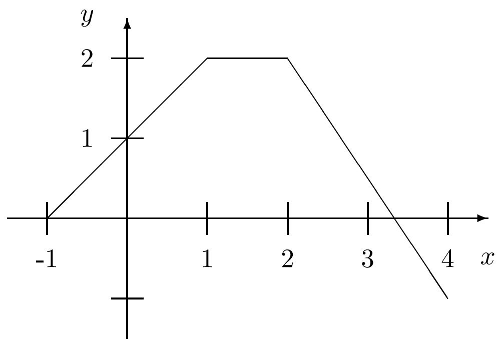
3. [10 points] Find the area of the finite region bounded between $y = 4 - x^2$ and $y = 4 - 2x$ for $0 \le x \le 2$, indicated below.



The upper function is $y = 4 - x^2$ while the lower function is $y = 4 - 2x$. Therefore, the area is

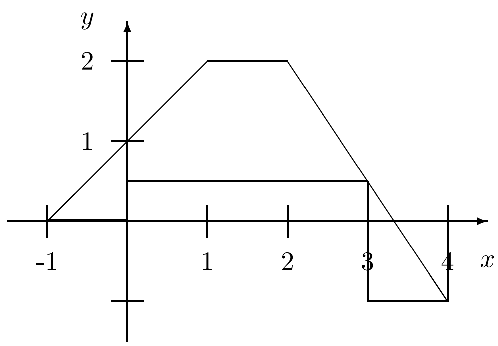
$$\begin{aligned}\int_0^2 \big((4-x^2) - (4-2x)\big)\,dx &= \int_0^2 (2x - x^2)\,dx \\ &= \left(x^2 - \tfrac{1}{3}x^3\right)\Big|_0^2 \\ &= (4 - \tfrac{8}{3}]) - (0-0) = \tfrac{4}{3}\end{aligned}$$

4. [16 points] A $f(x)$ is drawn below. It's graph consists of three line segments. Let P be the partition $P = \{-1, 0, 3, 4\}$.



a. On the graph of $f(x)$ above, draw the rectangles corresponding to $L_f(P)$.

The partition makes three subintervals $[-1,0]$, $[0,3]$, and $[3,4]$. On $[-1,0]$ the minimum value of the function is 0, so the first rectangle is a degenerate rectangle of height 0. On $[0,3]$ the minimum value of the function is 0.5, so that's the rectangle's height there. On $[3,4]$ the minimum value of the function is -1 so the rectangle has one side at $y=-1$.



b. Compute $L_f(P)$, and $U_f(P)$.

$L_F(P)$ is the total signed area of the rectangles you drew in part a. Except for the last rectangle, the tops are all above the x-axis, so only the last rectangle contributes a negative value.

$$L_F(P) = (0-(-1))0 + (3-0)0.5 + (4-3)(-1) = 0.5$$

The heights of the rectangles for $U_F(P)$ are 1, 2, and 0.5, so

$$U_F(P) = (0-(-1))1 + (3-0)2 + (4-3)0.5 = 7.5$$

c. Compute $\displaystyle\int_{-1}^{4} f(x)\,dx$

It's probably best to split the interval according to the three subintervals $[-1,1]$, $[1,2]$, and $[2,4]$. On the first interval we have $\displaystyle\int_{-1}^{1} f(x)\,dx = 2$ since it's the area of a triangle. On the second interval we have $\displaystyle\int_{1}^{2} f(x)\,dx = 2$ since it's the area of a rectangle. On the third interval, either you can integrate the function $\displaystyle\int_{2}^{4} (5-\tfrac{3}{2}x)\,dx$ or you can work with areas of triangles to get 1. Thus, the integral we're looking for is $2+2+1=5$.

5. [12 points] Suppose $f(x)$ and $g(x)$ are continuous functions defined on $[0,5]$ with $\displaystyle\int_0^3 f(x)\,dx = 5$, $\displaystyle\int_0^5 f(x)\,dx = 2$ and $\displaystyle\int_3^5 (3g(x)+1)\,dx = 20$.

a. What is $\displaystyle\int_5^0 f(x)\,dx$?

It's the negation of the integral from 0 to 5, namely -2.

b. What is $\displaystyle\int_0^5 (2f(x)-1)\,dx$?

$$\begin{aligned}\int_0^5 (2f(x)-1)\,dx &= 2\int_0^5 f(x)\,dx - \left(x\Big|_0^5\right)\\ &= 2\cdot 2 - 5 = -1\end{aligned}$$

c. What is $\displaystyle\int_3^5 g(x)\,dx$?

We know $\displaystyle\int_3^5 (3g(x)+1)\,dx = 20$, so $3\displaystyle\int_3^5 g(x)\,dx + \int_3^5 1\,dx = 20$. But $\displaystyle\int_3^5 1\,dx = 2$, therefore $3\displaystyle\int_3^5 g(x)\,dx = 18$. Thus, $\displaystyle\int_3^5 g(x)\,dx = 6$.

d. What is $\displaystyle\int_3^5 (f(x)+g(x))\,dx$?

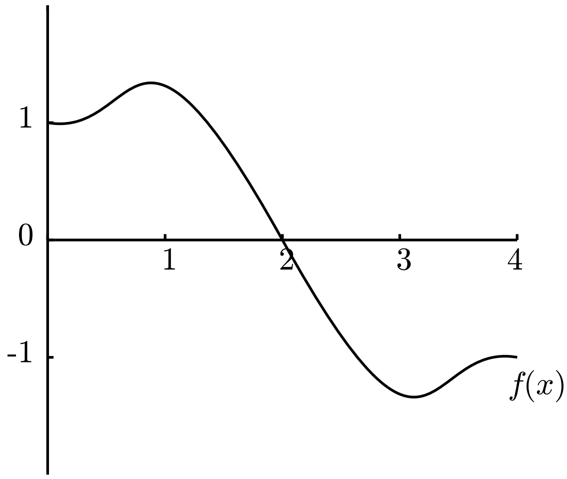
We already know $\displaystyle\int_3^5 g(x)\,dx = 6$. And

$$\int_3^5 f(x)\,dx = \int_0^5 f(x)\,dx - \int_0^3 f(x)\,dx = 2-5 = -3,$$

so $\displaystyle\int_3^5 (f(x)+g(x))\,dx = -3+6 = 3$.

6. [12 points] The function $f(x)$ is pictured below. Let $F(x) = \displaystyle\int_0^x f(t)\,dt$ be signed area under the curve from 0 to x, so

$$F(x) = \int_0^x f(t)\,dt.$$



For each of the following statements, say whether it is true or false, and give a brief reason why.

a. $F(2) > 1$. True. The curve is above the x-axis on $[0, 2]$ and that area includes a 1×1 square on the interval $[0, 1]$ and more.

b. $F(4) > 1$. False. In fact $F(4)$ looks like it's 0 since the area above the axis on $[0, 2]$ looks like it's exactly equal to the area below the axis on $[2, 4]$.

c. $F(0) = 1$. False. The integral $\int_0^0 f(t)\,dt$ is 0.

d. $F'(3) = 0$. False. $F'(3)$ is just $f(3)$ which is some negative value, about -1.3. (However, it looks like f has a minimum at $x = 3$ which says the f' is 0 there, so $F''(3)$ could be 0.)