Math 121, Calculus II
Clark University
Final Exam Answers, Spring 2009

1. [12] Suppose a point moves along the x-axis with acceleration

$$a(t) = \tfrac{1}{2}(e^{3t} - e^{-3t})$$

meters per second squared. If the point starts at the origin with velocity 30 meters per second, what is it's position 5 seconds later.

You can integrate the acceleration to find the velocity $v(t)$.

$$v(t) = \int \tfrac{1}{2}(e^{3t} - e^{-3t})\,dt = \tfrac{1}{6}(e^{3t} + e^{-3t}) + C.$$

The initial velocity is $v(0) = 30$, so $30 = \frac{1}{6}(e^0 + e^0) + C$. So $C = 30 - \frac{1}{3} = \frac{89}{3}$. Thus

$$v(t) = \tfrac{1}{6}(e^{3t} + e^{-3t}) + \tfrac{89}{3}.$$

You can integrate the velocity to find the position $f(t)$.

$$\begin{aligned} f(t) &= \int \left(\tfrac{1}{6}(e^{3t} + e^{-3t}) + \tfrac{89}{3}\right)\,dt \\ &= \tfrac{1}{18}(e^{3t} - e^{-3t}) + \tfrac{89}{3}\,t + C \end{aligned}$$

The initial position is $f(0) = 0$, so $0 = \frac{1}{18}(e^0 - e^0) + 0 + C$, so $C = 0$ this time. That gives the position at $t = 5$ to be $f(5) = \frac{1}{18}(e^{15} - e^{-15}) + \frac{89}{3}\cdot 5$.

2. [16] Suppose that

$$\begin{aligned} \int_0^2 (2f(x) + 3g(x))\,dx &= 5, \text{ and} \\ \int_0^2 (5f(x) + 2g(x))\,dx &= 4. \end{aligned}$$

a. Find $\displaystyle\int_0^2 (f(x) - g(x))\,dx$.

There are various ways you can figure out how to combine the two given equations to determine the value of that integral. Here's one way.

Let $F = \int_0^2 f(x)\,dx$ and $G = \int_0^2 g(x)\,dx$. Then we know that

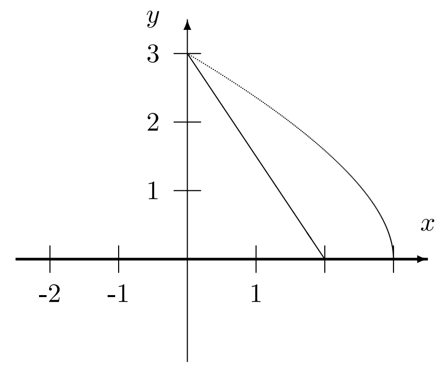
$$\begin{aligned} 2F + 3G &= 5 \\ 5F + 2G &= 4 \end{aligned}$$

You can then solve this system of simultaneous linear equations to determine that $F = \frac{2}{11}$ and $G = \frac{17}{11}$. Then $\int_0^2 (f(x) - g(x))\,dx = F - G = -\frac{15}{11}$.

b. What is the average value of $f(x)$ on the interval $[0, 2]$?

Since $\int_0^2 f(x)\,dx = \frac{2}{11}$ and the length of the interval is 2, therefore the average value of f is $\frac{1}{11}$.

3. [12] Find the area bounded by the curves $y = 0$, $3x+2y = 6$ and $3x + y^2 = 9$.



Note that the equations $3x + 2y = 6$ and $3x + y^2 = 9$ can either be written as functions of x or as functions of y. As functions of x they give $y = 3 - \frac{3}{2}x$ and $y = \sqrt{9 - 3x}$. But as functions of y they give $x = 2 - \frac{2}{3}y$ and $x = 3 - \frac{1}{3}y^2$.

Treating x as the independent variable gives the area as the sum of the integrals

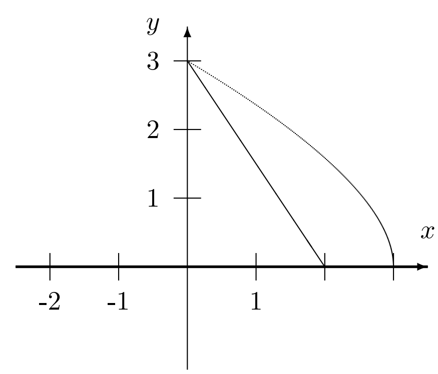
$$\int_0^2 \left(\sqrt{9 - 3x} - (3 - \tfrac{3}{2}x)\right)\,dx + \int_2^3 \sqrt{9 - 3x}\,dx,$$

but treating y as the independent variable gives

$$\int_0^3 \left((3 - \tfrac{1}{3}y^2) - (2 - \tfrac{2}{3}y)\right)\,dy.$$

Either way, you'll find that the area of the region is 3.

4. [12] Give an integral which expresses the volume of the solid of revolution obtained by rotation the region of problem 3 about the x-axis. The integral may be with respect to x or y. You need not evaluate the integral.



We'll look at both ways you could do it.

First, with respect to x. Since vertical cross sections of the plane region are being rotated about the x-axis, they will generate washers or disks. Thus, the volume is the sum of the integrals

$$\int_0^2 \left(\pi(\sqrt{9-3x})^2 - \pi(3-\tfrac{3}{2}x)^2\right)\,dx + \int_2^3 \pi(\sqrt{9-3x})^2\,dx.$$

Second, with respect to y. Horizontal cross sections will generate shells, so the volume is given by the integral

$$\int_0^3 2\pi y\left((3-\tfrac{1}{3}y^2)-(2-\tfrac{2}{3}y)\right)\,dy.$$

5. [16] Evaluate the following derivatives.

a. $\dfrac{d}{dx}\left(e^{3x^2+2}\ln(3x^2+2)\right)$

By the product rule, this is equal to

$$e^{3x^2+2}\cdot 6x\cdot\ln(3x^2+2) + e^{3x^2+2}\frac{1}{3x^2+2}6x$$

which you can simplify if you want

$$6xe^{3x^2+2}\left(\ln(3x^2+2)+\frac{1}{3x^2+2}\right)$$

b. $\dfrac{d}{dx}\left[\dfrac{\arctan x}{1+x^2}\right]$

By the quotient rule this equals

$$\frac{\frac{1}{1+x^2}(1+x^2)-(\arctan x)2x}{(1+x^2)^2}$$

which simplifies to $\dfrac{1-2x\arctan x}{(1+x^2)^2}$.

c. $\dfrac{d}{dx}\left[\dfrac{1+\ln x}{x}\right]$

Using the quotient rule you'll find $\dfrac{\frac{1}{x}\cdot x-(1+\ln x)\cdot 1}{x^2}$ which simplifies to $\dfrac{-\ln x}{x^2}$.

6. [32] Compute the following integrals:

a. $\displaystyle\int (e^2)^{1+x}\,dx$

One method to compute this integral is to simplify it first to $\int e^{2+2x}\,dx$. You can then make a substitution $u=2+2x$, or perhaps see right away that the integral equals $\frac{1}{2}e^{2+2x}+C$.

b. $\displaystyle\int \frac{\sin 3x}{3+\cos 3x}\,dx$

There are various substitutions that work here. The best is to take u to be the entire denominator so that $du = -3\sin 3x\,dx$. Then the integral transforms to $-\frac{1}{3}\int\frac{1}{u}\,du$ which equals $-\frac{1}{3}\ln u + C = -\frac{1}{3}\ln(3+\cos 3x)+C$.

c. $\displaystyle\int xe^{2x}\,dx$

This integral is designed for integration by parts with $u=x$ and $dv=e^{2x}\,dx$, so then $du=dx$ and $v=\frac{1}{2}e^{2x}$. Then you get $\frac{1}{2}xe^{2x}-\int\frac{1}{2}e^{2x}\,dx$ which equals $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$.

d. $\displaystyle\int xe^{x^2}\,dx$

The easiest way to compute this integral is to make a substitution with $u=x^2$ and $du=2\,dx$. Then the integral becomes $\frac{1}{2}\int e^u\,du$ which integrates as $\frac{1}{2}e^u+C=\frac{1}{2}e^{x^2}+C$.