The Natural Exponential Function

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The definition. The natural logarithm function $\ln x$ is an increasing function whose domain is the open interval $(0, \infty)$ and whose range is $\mathbf{R}$, all of the real numbers. Since it's increasing, therefore it's one-to-one and has an inverse, the *natural exponential function*, which we denote $\exp x$. Soon we'll also denote it e^x, but as that notation already has a meaning when x is a rational number, we'll have to show $\exp x$ agrees with e^x for rational x.

Now, exp is inverse to ln, and that means

$$\exp x = y \quad \text{iff} \quad x = \ln y.$$

Let's start by showing $\exp(m/n) = e^{m/n}$ for rational numbers m/n so we can use the usual exponential notation. We know that

$$\ln(e^{m/n}) = \frac{m}{n} \ln e = \frac{m}{n}.$$

But since exp is inverse to ln, that statement is equivalent to

$$e^{m/n} = \exp \frac{m}{n},$$

which is what we wanted to show.

Now we can use the usual exponential notation, but be aware that exp is still used when the exponent is complicated. For instance, a notation like $\exp \dfrac{(t-\mu_0)^2}{2\sigma_0/\sqrt{n}}$ is preferred by many over $e^{(t-\mu_0)^2/(2\sigma_0/\sqrt{n})}$.

Properties of the natural exponential function. That the natural exponential function is inverse to the natural log now reads

$$\boxed{e^x = y \quad \text{iff} \quad x = \ln y}$$

From this logical equivalence, every statement about ln can be converted to a statement about e^x.

Here's a table with properties of ln and exp.

$\text{Domain}(\ln) = (0, \infty)$	$\text{Range}(\exp) = (0, \infty)$
$\text{Range}(\ln) = \mathbf{R}$	$\text{Domain}(\exp) = \mathbf{R}$
$\lim_{x\to\infty} \ln x = \infty$	$\lim_{x\to\infty} e^x = \infty$
$\lim_{x\to 0^+} \ln x = -\infty$	$\lim_{x\to -\infty} e^x = 0$
$\ln 1 = 0$	$e^0 = 1$
$\ln e = 1$	$e^1 = e$
$\ln xy = \ln x + \ln y$	$e^{x+y} = e^x e^y$
$\ln \frac{x}{y} = \ln x - \ln y$	$e^{x-y} = \frac{e^x}{e^y}$
$\ln \frac{1}{y} = -\ln y$	$e^{-y} = \frac{1}{e^y}$
$\ln x^{m/n} = \frac{m}{n} \ln x$	$(e^x)^{m/n} = e^{xm/n}$

The first few lines in the table are direct translations of the properties of ln into properties of exp. The later ones are almost direct, but not quite. For example, let's see how the identity $\ln xy = \ln x + \ln y$ implies the identity $e^{x+y} = e^x e^y$. It would help to change the variables in one of the identities since they don't exactly correspond. Let's show the identity $\ln xy = \ln x + \ln y$ implies the identity $e^{s+t} = e^s e^t$.

Let $s = \ln x$ so that $x = e^s$, and let $t = \ln y$ so that $y = e^t$. Since $\ln xy = \ln x + \ln y$, therefore $s + t = \ln x + \ln y = \ln xy$. That implies $e^{s+t} = xy$, which equals $e^s e^t$, and that was what we wanted to show.

Note that the final equation in the right column of the table, $(e^x)^{m/n} = e^{xm/n}$ is not quite satisfactory. We would prefer to have $(e^x)^y = e^{xy}$ for any real number y, not just for rational $y = m/n$, but we haven't yet defined what $(e^x)^y$ would mean since we have only defined exponentiation when the base is e, not when the base is an arbitrary real number like e^x. That comes soon.

The derivative and integral of the natural exponential function. Recall that, for positive x, $\frac{d}{dx} \ln x = 1/x$, that is, $\int dx/x = \ln x + C$. Let's use that information to determine the derivative and integral of e^x. We'll use the inverse function theorem for derivatives to to that.

The inverse function theorem says that if $y = f(x)$ so that $x = f^{-1}(y)$, then

$$\frac{dx}{dy} = \frac{1}{dy/dx},$$

that is,

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

Right now, we have $y = f(x) = \ln x$, $x = f^{-1}(y) = e^y$, and $f'(x) = 1/x$. We want to find the derivative of e^y, that is, $\frac{d}{dy} e^y = dx/dy = (f^{-1})'(y)$. By the inverse function theorem, that equals

$$\frac{1}{dx/dy} = \frac{1}{f'(f^{-1}(y)} = \frac{1}{1/x} = x = e^y.$$

Thus, the derivative $\dfrac{d}{dy}e^y$ of the exponential function e^y is itself e^y. If we switch to x as the independent variable, we can write this result as the exponential rule for derivatives

$$\boxed{\frac{d}{dx}\,e^x = e^x}$$

It is this property of the exponential function that gives it lots of applications in science.

Frequently, an exponential function has an exponent that is not just the variable x, but a function of x, for instance e^{x^2+5x-3}. To find its derivative, use the exponential rule for derivatives along with the chain rule: $\frac{d}{dx}e^{x^2+5x-3} = e^{x^2+5x-3}\,\frac{d}{dx}(x^2+5x-3) = e^{x^2+5x-3}\,(2x+5)$. More generally,

$$\boxed{\frac{d}{dx}\,e^u = e^u\,\frac{du}{dx}}$$

We can immediately use the exponential rule for derivatives, $\frac{d}{dx}\,e^x = e^x$, to integrate the exponential function.

$$\boxed{\int e^x\,dx = e^x + C}$$