The Natural Logarithm

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You probably have studied exponents and logs for quite a while. First you understood the power x^n where n is a positive integer as repeated multiplication. Second, when $-n$ is negative integer, you took x^{-n} as the reciprocal of x^n, so long as x was not 0. And, of course, you took x^0 to be 1, again when x was not 0. (Note that 0^0 is not defined. It's one of those "indefinite forms" like 0/0.)

Next, you recognized $x^{1/n}$ as an n^{th} root, $\sqrt[n]{x}$, when x was a positive integer. And then $x^{m/n}$ was the m^{th} power of an n^{th} root. The usual proprieties of exponents x^y work just fine so long as the base x is positive and the the power y is a rational number m/n.

There are various different ways to extend the definition of exponents to irrational values of the exponent y. One method is to use principles of continuity to define x^y for any positive values of x and y. After that, the logarithm to the base b, $\log_b y$ can be defined to be the function inverse to the exponential function b^x, that is, $x = \log_b y$ iff $y = b^x$.

That approach to exponents and logs works okay, but we'll do it a different way. We'll use integrals to define logarithms first, then we'll define exponential functions to be inverse to logarithmic functions. It's really a very nice way to study logs and exponents, and it uses the theory we've developed this semester.

In this note, we'll do the first step, defining the natural logarithm.

The natural logarithm function. Recall that the power rule for integrals,

$$\int x^n \, dx = \tfrac{1}{n+1} \, x^{n+1} + C$$

only works when n does not equal -1. What happens when n does equal -1? That is, what is the integral of $1/x$? The graph of the funcion $1/x$, that is, the graph of the equation $y = 1/x$ is just the standard rectangular hyperbola. The function $1/x$ is continuous, except at $x = 0$ where it's not defined. We'll restrict our attention to where x is positive, then $1/x$ is continuous. Since it's continuous, it's integrable, and we know by the FTC^{-1} that

$$\int_a^x \frac{1}{t} \, dt$$

has the derivative $1/x$ so long as a is a (positive) constant. This integral gives the area under the hyperbola $y = 1/x$ between a and x.

We'll take a to be equal to 1 and give that function a name, *the natural logarithm.* In historical works you'll find it denoted $\log_e x$ to distinguish it from $\log_{10} x$, which is the logarithm to the base 10. Most mathematical works now simply denote it $\log x$, but for some reason calculus texts usually denote it $\ln x$, so that's what I'll do for the rest of this note. Thus, we define

$$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt}$$

for $x > 0$. Note that logs are not defined when x is negative or 0.

Properties of the natural logarithm. We designed the natural log so that its derivative is $1/x$, so that's our first property.

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

Very often, we need to compute the derivative of the log of something more complicated than x, and so we need the chain rule. Combining our first property with the chain rule we have

$$\boxed{\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}}$$

The next property comes directly from the definition, namely

$$\boxed{\ln 1 = 0}$$

since the integral from 1 to 1 of anything is 0.

Historically the most important properties of logs are that they turn multiplication into addition, division into subtraction, powers into multiplication, and roots into division. Thus, with a log table, if you know only how to add, subtract, and look things up, you can fairly easily perform all these operations. Of course, with the advent of the hand-held calculator, we have an even easier way to perform these operations. Time marches on. Although these properties aren't important for calculation anymore, they're still used all the time in science and mathematics.

Let's prove the first one.

Theorem. If x and y are positive, then

$$\boxed{\ln xy = \ln x + \ln y}$$

Proof. Treat y as a constant and x as a variable. Each side of the above equation is a function of x. The left hand side is $f(x) = \ln xy$ and the right hand side is $g(x) = \ln x + \ln y$. To show that the two functions are the same, we'll show first that they have the same derivative, that is, we'll show $f'(x) = g'(x)$. That will imply that the two functions differ by a constant. Next, we'll show that for some value of x that $f(x) = g(x)$. That will imply that the constant is 0, so the two functions are equal, that is, the equation is always true.

Step 1. We'll need the chain rule for the derivative of the left hand side.

$$f'(x) = \frac{d}{dx} \ln xy = \frac{1}{xy} \frac{d}{dx} xy = \frac{1}{xy}\, y = \frac{1}{x}.$$

The derivative of the right hand side is easier. $g'(x) = \frac{1}{x} + 0 = \frac{1}{x}$. Thus, they have the same derivative and so differ by a constant.

Step 2. Take $x = 1$. Then $f(1) = \ln 1y = \ln y$, and $g(1) = \ln 1 + \ln y = 0 + \ln y = \ln y$. Thus, f and g agree for some value of x. Therefore the constant they differ by is 0, and they are equal. Q.E.D.

We could prove the next property, that logs convert division into subtraction the same way, but that follows more easily from the preceding theorem.

Theorem. If x and y are positive, then

$$\boxed{\ln \frac{x}{y} = \ln x - \ln y}$$

Note in particular that

$$\boxed{\ln \frac{1}{y} = -\ln y}$$

Proof. Let $z = x/y$. Then $x = yz$, so $\ln x = \ln y + \ln z$. Hence, $\ln z = \ln x - \ln y$, that is, $\ln x/y = \ln x - \ln y$. Q.E.D.

So far, we haven't really defined x^y for all positive values of y, just when y is a rational number m/n. So, we'll restrict powers to rational numbers for now, but later we will define x^y for all positive y.

Theorem. If x is positive and m/n is a rational number, then

$$\boxed{\ln x^{m/n} = \frac{m}{n} \ln x}$$

Note in particular that

$$\boxed{\ln x^m = m \ln x} \quad \text{and} \quad \boxed{\ln \sqrt[n]{x} = \frac{1}{n} \ln x}$$

Proof. Again, we'll show first that the derivatives of the two sides of the equation are equal, and second that the two sides agree for some value of x.

We can calculate the derivative of the left hand side with the help of the chain rule

$$\frac{d}{dx} \ln x^{m/n} = \frac{1}{x^{m/n}} \frac{d}{dx} x^{m/n} = \frac{1}{x^{m/n}} \frac{m}{n} x^{m/n-1} = \frac{m}{nx}.$$

The derivative of the right hand side is easier. $\frac{d}{dx} \frac{m}{n} \ln x = \frac{m}{nx}$. Thus, the two sides have the same derivative.

Now, if we take x to be 1, then $\ln 1^{m/n}$ does, indeed, equal $\frac{m}{n} \ln 1$, since both are 0. Q.E.D.

The range and growth of the natural logarithm. I'm going to skip the proof of the next theorem and just quote it instead. It says that the range of $\ln x$ is all of the real numbers. Moreover, it's an increasing function, so it's one-to-one, and

$$\boxed{\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to 0^+} \ln x = -\infty}$$

Although $\ln x$ grows to infinity as you let x grow to infinity, it actually grows extremely slowly, slower than $\sqrt{x}$ or any other fractional power of x.

Likewise, if you let x decrease to 0, $\ln x$ approaches $-\infty$ extremely fast. Thus, the graph $y = \ln x$ is asymptotic to the y-axis in the negative direction.

The number e, the base of the natural logarithm. Our definition of $\ln x$ allows us to define e very easily. Since $\ln x$ takes on all real numbers as values, in particular it takes on the number 1. We define e to be that number such that

$$\boxed{\ln e = 1}$$

In other words, under the hyperbola $y = 1/x$ between $x = 1$ and $x = e$ the area is 1.

When x is negative. So far, we know the derivative of $\ln x$ is $1/x$, so the integral of $1/x$ is $\ln x + C$. But that's only valid when x is positive. When x is negative $1/x$ is continuous, so it has an integral there, too. What is it?

It turns out it's just $\ln(-x)$. Since $-x$ is positive, $\ln(-x)$ is defined, and by the chain rule,

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}\,\frac{d}{dx}(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus, when x is negative

$$\int \frac{1}{x}\,dx = \ln(-x) + C.$$

We can combine the two cases, when x is positive $\int \frac{1}{x}\,dx = \ln x + C$ while when x is negative $\int \frac{1}{x}\,dx = \ln(-x) + C$, into one by using absolute values

$$\boxed{\int \frac{1}{x} = \ln|x| + C}$$

but you have to be careful that for this to be true, x cannot be 0 but has to be always positive or always negative.