Substitution for Integrals

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We've looked at the basic rules of integration and the Fundamental Theorem of Calculus (FTC). Unlike differentiation, there are no product, quotient, and chain rules for integration. But, the product rule and chain rule for differentiation do give us the two most important techniques of integration, namely the techniques called *integration by parts* and *substitution*. We'll first look at substitution, and later at integration by parts.

Leibniz' notation. One of the main reasons we continue to use Leibniz' notation in calculus is that it works so well, even though we usually don't use the infinitesimals (infinitely small quantities) that Leibniz used. For differentiation, we often write $\dfrac{dy}{dx}$ for the derivative of y with respect to x because that notation indicates the dependent variable y and the independent variable x. But for Leibniz, the derivative $\dfrac{dy}{dx}$ was an actual quotient, namely, the infinitesimal change dy, which he called the differential of y, divided by the infinitesimal change dx, the differential of x.

With that notation, the chain rule is simply stated as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

That's a lot simpler expression than what we get when we use functional notation:

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

The connection is that $y = f(u)$ and $u = g(x)$.

Leibniz interpreted integrals as infinite sums of infinitesimals. When he wrote something like $\displaystyle\int_3^5 x^2\,dx$, he meant to sum all the quantities $x^2\,dx$ for x taking values from 3 to 5, where dx was the differential for that value of x, that is, an infinitely small change in x. We continue to use Leibniz' notation for integrals because it works so well for the technique of substitution.

An example. Let's start with an example with an indefinite integral so we don't have to worry about the limits of integration. Since the chain rule is about composition of functions, our example should involve composition. Consider the integral

$$\int x\cos(3x^2+5)\,dx.$$

This is one you can figure out by inspection that the answer is going to be $\frac{1}{6}\sin(3x^2+5)+C$, but let's suppose we didn't notice that.

We do notice that there is a composition of functions, namely the cosine of a function, the function $u = g(x) = 3x^2 + 5$. And we know the derivative of that function, namely, $\dfrac{du}{dx} = g'(x) = 6x$. Following Leibniz, the derivative $\dfrac{du}{dx}$ is a quotient, so we will rewrite that last equation as $du = 6x\,dx$. In the integral we're looking at, we have the a factor of $x\,dx$, and that's equal to $\frac{1}{6}du$. Thus, we'll rewrite the integral as

$$\int \tfrac{1}{6} \cos u\, du.$$

That's easy to integrate, and we get $\frac{1}{6} \sin u + C$. We convert back to the original variable x, and we get our answer $\frac{1}{6} \sin(3x^2 + 5) + C$.

That's all there is to it, but it probably looks right now more like magic than like mathematics. It would be nice to have a rigorous proof that it works. Here's the theorem for definite integrals. If you leave out the limits of integration, you'll see the theorem for indefinite integrals.

Theorem. Let $y = f(u)$ and $u = g(x)$ where f is continuous and g is differentiable. Then

$$\int_a^b y\, \frac{du}{dx}\, dx = \int_{g(a)}^{g(b)} y\, du.$$

Proof. Let F be an antiderivative of f, that is, $F'(u) = f(u) = y$. Then the right integral equals $F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$.

By the chain rule,

$$(F \circ g)'(x) = F'(g(x))\, g'(x) = F'(u)\, \frac{du}{dx} = y\, \frac{du}{dx}.$$

Therefore, $F \circ g$ is an antiderivative of $y\,\dfrac{du}{dx}$, and so the left integral equals $(F \circ g)\Big|_a^b = F(g(b)) - F(g(a))$.

Thus, the two integrals are equal. Q.E.D.

An example. There are two ways you can evaluate a definite integral like

$$\int_0^2 x \cos(3x^2 + 5)\, dx$$

using the method of substitution.

One way is to temporarily forget the limits of integration and treat it as an indefinite integral. Make the substitution $u = 3x^2 + 5$ as done above to simplify the integral, do the integration in terms of u, back substitute to get the answer in terms of x. Then recall the limits $x = 0$ to $x = 2$, and evaluate $\frac{1}{6} \sin(3x^2 + 5)\Big|_0^2$ to get $\frac{1}{6} \sin 17 - \frac{1}{6} \sin 5$.

The easier way is to change the limits to be in terms of u at the same time the substitution is made as indicated in the statement of the theorem above. When $x = 0$, $u = 3 \cdot 0^2 + 5 = 5$ and when $x = 2$, $u = 3 \cdot 2^2 + 5l = 17$. Thus, the computations looks like

$$\begin{aligned} \int_{x=0}^2 x \cos(3x^2 + 5)\, dx &= \int_{u=5}^{17} \tfrac{1}{6} \cos u\, du \\ &= \tfrac{1}{6} \sin u\Big|_5^{17} \\ &= \tfrac{1}{6} \sin 17 - \tfrac{1}{6} \sin 5 \end{aligned}$$

The advantage to this method is that you don't have to return back to the original variable.

When do you use substitution? The usual purpose of substitution is to simplify an integral a bit. Look for a composition somewhere in the integrand and choose the inner function to be $u = g(x)$. But in order for the substitution to work, the derivative of the inner function has to also appear as a factor of the integrand, although constant multiples can be adjusted for. In the examples above, the integrand $x\cos(3x^2+5)$ had a composition with $u = g(x) = 3x^2+5$ as the inner function, and its derivative, $6x$, is a factor of the integrand, except for the constant multiple 6 that can be taken care of. Here are a couple of other examples.

In the integral $\int 14x^2\sqrt{5x^3-8}\,dx$, there's a composition where $u = 5x^3-8$ is the inner function, and its derivative, $15x^2$, is a factor of the integrand except for the constant multiple 15. So that substitution will work.

In the integral $\int 14x\sqrt{5x^3-8}\,dx$, the same substitution $u = 5x^3-8$ won't work since there's no factor of x^2 in the integrand.

In the integral $\int \frac{6\sin x}{\cos x}\,dx$, you can treat the denominator as the inner function $u = \cos x$, and its derivative $-\sin x$ is a factor of the integrand (except for the constant multiple of -1 which is not a problem), so this substitution works. But it wouldn't work if there were no $\sin x$ in the numerator.

In very complicated integrals, you may end up using substitution twice, first to simplify it partway, second to simplify it even more. Usually when you do two substitutions, you could have done one grand substitution instead, but that's not always easy to see at first.