Math 130 Linear Algebra

Prof. D. Joyce, Clark University

Wednesday, 9 Sep 2009

Due Today. Exercises from section 1.3: 1–4, 7, 9, 11–12, 19–20, 33, T1, T4.

Read for Friday section 1.5 on matrix transformations and start 1.6 which goes into the solutions of linear systems of equations in a little more detail than we already have.

Quiz Friday. Covering through section 1.3.

Due Monday. Exercises from section 1.4: 11–13, 19, T.10, T.24, and T.30. Also, there will be a quiz on Friday covering through section 1.3.

Last time. We introduced vectors and their dot products. We'll continue those notes today on matrix multiplication.

Today and next time. Properties of the matrix operations.

Properties of the matrix operations. The operations we're talking about are addition and subtraction of matrices, scalar multiplication (that is, the product of a scalar and a matrix), and multiplication of matrices. There is no division of matrices, but we will soon be very interested in inverting square matrices.

Addition and subtraction. Addition and subtraction of matrices is just like addition and subtraction of numbers. That's because they are done coordinatewise. So, for instance, addition is commutative, that is, $A+B=B+A$, and it's associative, that is, $(A+B)+C=A+(B+C)$. Also, for each shape matrix $(m \times n)$, there is a zero matrix where all the entries are 0, and that matrix acts like the additive identity, that is, $A+0=0+A=A$. Note that even though there are infinitely many zero matrices, one for each shape, we'll denote them all by the same symbol, namely 0 (although it looks like O in the text). Furthermore, subtraction has the usual properties, too, properties like $A-A=0$.

Scalar multiplication. Scalar multiplication works as you would expect, too. If r and s are scalars while A and B are matrices of the same shape, then

$$\begin{aligned} r(sA) &= (rs)A \\ (r \pm s)A &= rA \pm sA \\ r(A \pm B) &= ra \pm rB \end{aligned}$$

are all identities, and for the same reason, that is, all the operations are performed coordinatewise.

Matrix multiplication. We should expect matrix multiplication to be different. We already saw an example in the last section were it's not commutative, that is, where $AB \neq BA$. But, it turns out that multiplication *does* have many of the expected properties. It is associative, and it does distribute over addition and subtraction both on the left and on the right. (Note that since multiplication is not commutative, we have to be very careful about the order of multiplication). So when A, B, and C are the right shape matrices so the the operations can be performed, then the the following are always identities:

$$\begin{aligned} A(BC) &= (AB)C \\ A(B \pm C) &= AB \pm AC \\ (A \pm B)C &= AC \pm BC \end{aligned}$$

I'll give some arguments in class why these are always true. Associativity of multiplication hinges on the identity

$$\begin{aligned} & \sum_j a_{ij} \left(\sum_k b_{jk} c_{kl} \right) \\ = & \sum_{j,k} a_{ij} b_{jk} c_{kl} \\ = & \sum_k \left(\sum_j a_{ij} b_{jk} \right) c_{kl}. \end{aligned}$$

The identity matrices. Just like there are matrices that work as additive identities (we denoted them all 0 as described above), there are matrices that work as multiplicative identities, and we'll denote them all I and all them *identity matrices*. An identity matrix is a square n by n matrix with 1 down the diagonal and 0 elsewhere. In our text it's denoted I_n to emphasize its size, but you can always tell by the context what its size is, so I usually leave out the index n. By the way, whenever you've got a square n by n matrix, you can say the *order* of the matrix is n. Anyway, I acts like an identity matrix

$$AI = A = IA.$$

Note that if A is not a square matrix, then the orders of the two identity matrices I in the identity $AI = A = IA$ are different. For example,

$$\begin{aligned} & \begin{bmatrix} 4 & 5 & 6 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} 4 & 5 & 6 \\ 3 & -1 & 0 \end{bmatrix} \\ = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 3 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Cancellation doesn't work for matrix multiplication! Not only is matrix multiplication noncommutative, but the cancellation law doesn't hold for it. You're familiar with cancellation for numbers: if $xy = xz$ but $x \neq 0$, then $y = z$. But we can come up with matrices so that $AB = AC$ and $A \neq 0$, but $B \neq C$. For example $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

Powers of matrices. Frequently, we'll multiply square matrices by themselves (you can only multiply square matrices by themselves), and we'll use the standard notation for powers. The expression A^p stands for the product of p copies of A. Since matrix multiplication is associative, this definition works, so long as p is a positive integer. But we can extend the definition to $p = 0$ by making $A^0 = I$, and the usual properties will will still hold. That is, $A^pA^q = A^{p+q}$ and $(A^p)^q = A^{pq}$. Later, we'll extend powers to the case when A is an invertible matrix and the power p is a negative integer.

Warning: because matrix multiplication is not commutative in general, it is usually the case that $(AB)^p \neq A^pB^p$.

More on scalar matrices and scalars. I can now explain why a scalar matrix is called a scalar matrix. A scalar matrix with r down the diagonal is just the scalar r times the identity matrix. For instance,

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 4I.$$

Among other things, that means that we can identify a scalar matrix with the scalar.

Transposition. We'll be using transposition of matrices throughout the course, and it's easy to verify the following properties, except, perhaps, the one involving multiplication where the order of multiplication is reversed.

$$\begin{aligned}
(A \pm B)^T &= A^T \pm B^T \\
(rA)^T &= r(A^T) \\
(AB)^T &= B^T A^T
\end{aligned}$$

Symmetric matrices. One very useful kind of square matrix is a symmetric matrix. A *symmetric matrix* is a square matrix which equals its own transpose. That is, $A^T = A$. In terms of elements, every pair of elements a_{ij} and a_{ji} symmetrically across the main diagonal are equal, $a_{ij} = a_{ji}$. For instance,

$$\begin{bmatrix} 4 & -2 & 3 \\ -2 & 5 & -8 \\ 3 & -8 & 0 \end{bmatrix}$$

is a 3 by 3 symmetric matrix. Of course, all diagonal matrices are also symmetric matrices.

In some of the exercises, we'll look at matrices that are *skew symmetric.* They're matrices A such that $A^T = -A$.

Exercises in MATLAB. If we have time, we'll look as some of the MATLAB exercises at the end of section 1.4, in particular, 1, 3, 4, and 7.