Math 130 Linear Algebra

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Due Wednesday. Exercises from section 3.1. Do parts a and b only of all of these exercises: 1–6, 15, 19, and 20.

Due Friday. Exercises from section 3.1: T1, T6, and T12.

First test. Next Monday 5 Oct, to cover through section 3.1.

Last time. Introduction to determinants, determinants of small matrices, permutations, and definition of a determinant. Recall the determinant of an $n \times n$ matrix: First form all $n!$ products of n elements, one element chosen out of each row and column. Next, negate all those that correspond to odd permutations, but don't negate those that correspond to even permutations. Finally, add all these $n!$ terms together. That's the determinant.

Today. We need to look at some properties of determinants that will help us compute them. There are quite a number of properties of determinants. We'll prove only a few of them in class.

Transposing a matrix. It is not easy to prove it, but the determinant of the transpose of a matrix is the same as the determinant of the matrix itself.

It's clear that the $|A|$ and $|A^T|$ are computed using the same $n!$ terms, each term being a product of n elements, one element chosen out of each row and column. What isn't easy to show is that the sign, plus or minus, is the same for each. That depends on showing that the parity of the permutation is the same whether you're using which column to choose for each row or which row you're using for each column. We'll skip that proof.

Interchanging rows. If two rows of a matrix are interchanged, then the determinant is negated. That is to say, if A is the original matrix, and the matrix B results if rows r and s of A are interchanged, then $|B| = -|A|$.

Both determinants, $|B|$ and $|A|$, are formed from the same $N!$ terms, but if you examine the parity of the permutation of a term for $|B|$, it's the opposite of the term for $|A|$. You'll actually prove this yourself when you do exercise T1 from section 3.1.

Likewise, if two columns of a matrix are interchanged, then the determinant is also negated.

Equal rows. It follows directly from the last theorem that if two rows of a matrix are equal, then the determinant of that matrix is 0.

That's because if you exchange the two rows, the determinant is supposed to be negated, but the resulting matrix is the same as the original. That means that its determinant is its own negation. The only number which is its own negation is 0.

Likewise, if two columns in a matrix are the same, then its determinant is 0.

A zero row. If a matrix has all 0s in a row, then its determinant is 0.

That's because each of the $n!$ terms includes a 0 in the product, since one entry of that row has to be chosen for the product.

Likewise, if a matrix has a 0 column, then its determinant is 0.

Multiplying a row by a constant. If you multiply every element in a row of a matrix by the same number, then the determinant is multiplied by that number. That is, if every entry in one row of the matrix A is multiplied by the constant c to get the matrix B, then $|B| = c|A|$.

That's because each of the $n!$ terms for $|B|$ looks just like the corresponding term for $|A|$ except for a factor of c that appears with the entry for that particular row.

Likewise, if you multiply a column by a constant, then the determinant is multiplied by a constant.

Note that now we can generalize the "equal row" theorem above to say that if one row of a matrix is a multiple of another, then the determinant is 0. Likewise for columns.

Multilinearity of the determinant. Suppose that two matrices are identical in all rows but one, and a third matrix is created which is the same as the two in all rows but that one, and in that one, the entries are the sums of the entries of the two matrices. Then the determinant of the resulting matrix is the sum of the determinants of the two matrices.

For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

This works because each of the $n!$ terms is the sum of the corresponding term of the first matrix and the corresponding term of the second matrix. For instance, the first term $a_{11}(a_{22}+b_{22})a_{33}$ is the sum $a_{11}a_{22}a_{33} + a_{11}b_{22}a_{33}$.

Likewise for columns.

Adding a multiple of one row to another. If you change a matrix by adding a multiple of one row to another row, then the determinant doesn't change.

This follows from the multilinearity property described above. The new matrix is formed from the original matrix and another matrix where one row is a multiple of another row, hence has 0 determinant. For instance, if you add twice the first row to the second, you can interpret the resulting matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+2a_{11} & a_{22}+2a_{12} & a_{23}+2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

as being created from these two matrices with the all same rows but the middle one

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{11} & 2a_{12} & 2a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and the second matrix has a 0 determinant because one row is a multiple of another. Therefore, the resulting matrix has the same determinant as the first matrix.