Math 130 Linear Algebra

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Due today. Exercises from section 3.2: 1, 3ab, 4ab, 10ab, 11ab, 20, 23.

Monday. Fall break; no class.

Due Wednesday. Exercises from section 4.1: 3, 5, 9ab, 13, 19, 21ab, 22ab.

Last time. Discussed cofactor expansion of determinants and Cramer's rule.

Today. We'll begin chapter 4: vectors in the plane $\mathbf{R}^2$. We'll interpret vectors as arrows, get a correspondence between vectors and points, look at the length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$, study properties of dot products $\mathbf{v}\cdot\mathbf{w}$, see that the dot product $\mathbf{v}\cdot\mathbf{w}$ between two vectors is related to the cosine of the angle between them.

Vectors in the plane $\mathbf{R}^2$. Recall that the vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ can be interpreted as an arrow in the plane $\mathbf{R}^2$ with a certain length and a certain direction. Actually, it can be interpreted as lots of different arrows with that length and direction. When it's put in "standard position," the head of the arrow is at the point (x, y) and the tail of the arrow is at the origin $(0, 0)$. Using this standard position, we identify a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with the point (x, y). Thus, sometimes we say "the vector (x, y)" deliberately confusing the point (x, y) with the vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

Sometimes we want to think of the vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as an arrow somewhere else in the plane, but it will have to be an arrow parallel to the standard arrow and with the same length. Then it can be any arrow with head (c, d) and tail (a, b) so long as $c - a = x$ and $d - b = y$. We'll say the vector

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c-a \\ d-b \end{bmatrix}$$

is the *displacement vector* from the point (a, b) to the point (c, d).

The length $\|\mathbf{v}\|$ of a vector $\mathbf{v}$. By the Pythagorean theorem of plane geometry, the distance $\|(x, y)\|$ between the point (x, y) and the origin $(0, 0)$ is

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Thus, we define the *length* of a vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ as being

$$\|\mathbf{v}\| = \left\| \begin{matrix} x \\ y \end{matrix} \right\| = \sqrt{x^2 + y^2}.$$

Note that we can interpret the square of the length of the vector as a dot product. Since

$$\|\mathbf{v}\|^2 = \left\| \begin{matrix} x \\ y \end{matrix} \right\|^2 = x^2 + y^2,$$

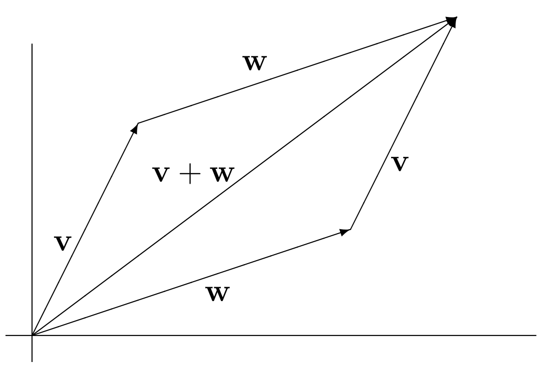
therefore $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$. Because of this connection between length and dot product, we can often reduce computations involving length to simpler computations involving dot products.

Incidentally, the length of a vector is sometimes called the *norm* of the vector, and it's sometimes denoted $|\mathbf{v}|$ instead of $\|\mathbf{v}\|$.

Geometric interpretation of addition of vectors. Recall that you add vectors coordinatewise, that is,

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} x+s \\ y+t \end{bmatrix}.$$

When you interpret vectors as arrows, there are a couple of ways to interpret addition. One is by what is called the "parallelogram law." In order to see the sum $\mathbf{v}+\mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$, draw both $\mathbf{v}$ and $\mathbf{w}$ in their standard positions with their tails at the origin. Then translate $\mathbf{v}$ so that its tail is at the head of $\mathbf{w}$ to get another arrow which can also be labelled $\mathbf{v}$. Likewise, translate $\mathbf{w}$ so that its tail is at the head of the original arrow $\mathbf{v}$ to get another arrow labelled $\mathbf{w}$. That completes a parallelogram with two sides being parallel $\mathbf{v}$ arrows and the other two sides parallel $\mathbf{w}$ arrows.



The arrow which goes from the origin to the opposite corner of the parallelogram is the vector $\mathbf{v}+\mathbf{w}$. Thus, you can see $\mathbf{v}+\mathbf{w}$ either as the arrow $\mathbf{v}$ followed by the arrow $\mathbf{w}$, or as the arrow $\mathbf{w}$ followed by the arrow $\mathbf{v}$.

Once we've got a geometric interpretation for addition, we automatically get geometric interpretations for negation and subtraction. The negation $-\mathbf{v}$ of a vector $\mathbf{v}$ is an arrow that points in the opposite direction of $\mathbf{v}$. We'll see a geometric interpretation of $\mathbf{w}-\mathbf{v}$ in a couple of paragraphs.

Properties of dot products $\mathbf{v}\cdot\mathbf{w}$. Recall that the dot product of two vectors is the sum of the products of corresponding elements, that is,

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} s \\ t \end{bmatrix} = xs + yt.$$

Dot products are often called *inner products* instead, and there are various other notations for them including $(\mathbf{u},\mathbf{v})$ and $(\mathbf{u}|\mathbf{v})$ and $\langle\mathbf{u}|\mathbf{v}\rangle$. We'll continue to use the terminology and notation of the text in this course.

The dot product acts like multiplication in a lot of ways, but not in all ways. First of all, the dot product of two vectors is a scalar, not another vector. That means you can't even ask if it's associative because the expression $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w}$ doesn't even make sense; $(\mathbf{u}\cdot\mathbf{v})$ is a scalar, so you can't take its dot product with the vector $\mathbf{w}$.

But aside from associativity, dot products act a lot like ordinary products. For instance, dot products are commutative:

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}.$$

Also, dot products distribute over addition,

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w},$$

and over subtraction,

$$\mathbf{u}\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w}.$$

Also, the zero vector $\mathbf{0}$ acts like zero:

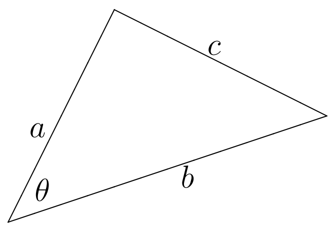
$$\mathbf{v}\cdot\mathbf{0} = 0.$$

Furthermore, dot products and scalar products have a kind of associativity, namely, if c is a scalar, then

$$(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v}).$$

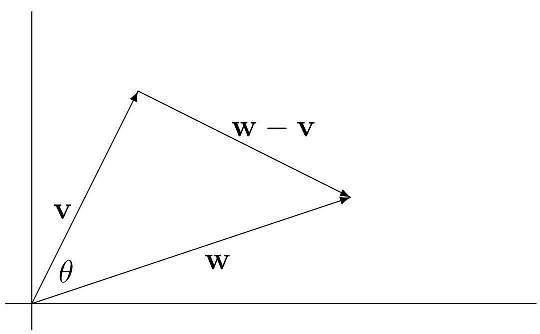
One more important property is that $\mathbf{v}\cdot\mathbf{v}$ is always greater than or equal to 0, but it only equals 0 when $\mathbf{v}=\mathbf{0}$. In terms of lengths of vectors, this property says $\|\mathbf{v}\| \geq 0$, and $\|\mathbf{v}\| = 0$ only when $\mathbf{v}=\mathbf{0}$.

The dot product $\mathbf{v}\cdot\mathbf{w}$ between two vectors and the cosine of the angle between them. The law of cosines for oblique triangles says that given a triangle with sides a, b, and c, and angle θ between sides a and b,



$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Now, start with two vectors $\mathbf{v}$ and $\mathbf{w}$, and place them in the plane with their tails at the same point. Let θ be the angle between these two vectors. The vector that joins the head of $\mathbf{v}$ to the head of $\mathbf{w}$ is $\mathbf{w}-\mathbf{v}$. Now we can use the law of cosines to see that



$$\|\mathbf{w}-\mathbf{v}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta.$$

We can convert the distances to dot products to simplify this equation.

$$\begin{aligned}\|\mathbf{w}-\mathbf{v}\|^2 &= (\mathbf{w}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v})\\ &= \mathbf{w}\cdot\mathbf{w} - 2\,\mathbf{w}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}\\ &= \|\mathbf{w}\|^2 - 2\,\mathbf{w}\cdot\mathbf{v} + \|\mathbf{v}\|^2\end{aligned}$$

Now, if we subtract $\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$ from both sides of our equation, and then divide by -2, we get

$$\mathbf{w}\cdot\mathbf{v} = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta.$$

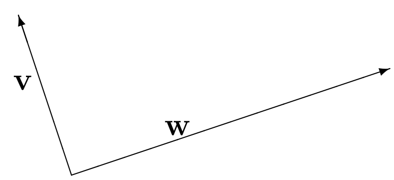
That gives us a way of geometrically interpreting the dot product. We can also solve the last equation for $\cos\theta$,

$$\cos\theta = \frac{\mathbf{w}\cdot\mathbf{v}}{\|\mathbf{v}\|\,\|\mathbf{w}\|},$$

which will allow us to do trigonometry by means of linear algebra. Note that

$$\theta = \arccos\left(\frac{\mathbf{w}\cdot\mathbf{v}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right).$$

Orthogonal vectors. The word "orthogonal" is synonymous with the word "perpendicular," but for some reason is preferred in many branches of mathematics. We'll write $\mathbf{w}\perp\mathbf{v}$ if the vectors $\mathbf{w}$ and $\mathbf{v}$ are orthogonal, or perpendicular.



Two vectors are orthogonal if the angle between them is 90°. Since the cosine of 90° is 0, that means

$$\mathbf{w}\perp\mathbf{v} \text{ if and only if } \mathbf{w}\cdot\mathbf{v} = 0$$

Vectors in Matlab. You can easily find the length of a vector in Matlab; where the length of a vector is called its `norm`. Let's find the length of two vectors and the angle between them using the formula

$$\theta = \arccos\left(\frac{\mathbf{w}\cdot\mathbf{v}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right).$$

Note that arccosines are computed with the `acos` function.

```
>> u = [3 4]

u =
```

```
     3     4

>> norm(u)

ans =
     5

>> v = [5 12]

v =
     5    12

>> norm (v)

ans =
    13

>> dot(u,v)

ans =
    63

>> costheta = dot(u,v)/(norm(u)*norm(v))

costheta =
    0.9692

>> acos(costheta)

ans =
    0.2487
```

Thus, the angle between the vectors $(3, 4)$ and $(5, 12)$ is 0.2487 radians.