Math 130 Linear Algebra

Prof. D. Joyce, Clark University

Friday, 23 Oct 2009

Quiz today.

Due Monday. Exercises from section 4.3: 1, 4–7, 21, 22.

Due Wednesday. From section 4.3: 25, 26, 27, 29, T9, T10, T11.

Today. We'll start section 5.1, Cross Products in $\mathbf{R}^3$: their definition, properties, and length; and triple scalar products.

The definition of cross products. The cross product $\times : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ is an operation that takes two vectors $\mathbf{u}$ and $\mathbf{v}$ in space and determines another vector $\mathbf{u} \times \mathbf{v}$ in space. (Cross products are sometimes called outer products, sometimes called vector products.) Although we'll define $\mathbf{u} \times \mathbf{v}$ algebraically, its geometric meaning is understandable. The vector $\mathbf{u} \times \mathbf{v}$ will have a length equal to the area of the parallelgram whose sides are $\mathbf{u}$ and $\mathbf{v}$, and the direction of $\mathbf{u} \times \mathbf{v}$ will be orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$ in a direction determined by a right-hand rule (when the coordinate system is right-handed).

The easiest way to define cross products is to use the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for $\mathbf{R}^3$. If

$$\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k},$$

and

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

then $\mathbf{u} \times \mathbf{v}$ is defined as

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

which is much easier to remember when you write it as a determinant

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

Properties of cross products. There are a whole lot of properties that follow from this definition. First of all, it's anticommutative

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}),$$

so any vector cross itself is $\mathbf{0}$

$$\mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

It's bilinear, that is, linear in each argument, so it distributes over addition and subtraction, $\mathbf{0}$ acts as zero should, and you can pass scalars in and out of arguments

$$\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})$$

$$(\mathbf{u} \pm \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \pm (\mathbf{v} \times \mathbf{w})$$

$$\mathbf{0} \times \mathbf{v} = \mathbf{0} = \mathbf{v} \times \mathbf{0}$$

$$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$$

Cross product is not associative, but we do have

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$$

These last two identities aren't so important, but we'll use them later today.

A couple more properties you can check from the definition, or from the properties already found are that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = 0$ and $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$. Those imply that the vector $\mathbf{u}\times\mathbf{v}$ is orthogonal to both vectors $\mathbf{u}$ and $\mathbf{v}$, and so it is orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$.

Standard unit vectors and cross products. Interesting things happen when we look specifically at the cross products of standard unit vectors. Of course

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0},$$

since any vector cross itself is $\mathbf{0}$. But

$$\mathbf{i}\times\mathbf{j} = \mathbf{k}, \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j},$$

and

$$\mathbf{j}\times\mathbf{i} = -\mathbf{k}, \qquad \mathbf{k}\times\mathbf{j} = -\mathbf{i}, \qquad \mathbf{i}\times\mathbf{k} = -\mathbf{j},$$

all of which follows directly from the definition.

Triple scalar products. Sometimes a "triple scalar product" of three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is defined as the determinant

$$[\mathbf{u},\mathbf{v},\mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Note that the triple product $[\mathbf{u},\mathbf{v},\mathbf{w}]$ is a scalar, not a vector. Triple scalar products can be written in terms of cross and dots products in several ways including

$$[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}).$$

It turns out that the absolute value of this triple product is the volume of the parallelepiped whose edges are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Length of the cross product. We'll compute the length $\|\mathbf{u}\times\mathbf{v}\|$ and show it equals $\|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta$ where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Each of the following equalities comes from properties of dot product or one of the statements mentioned above about cross product.

$$\begin{aligned}
\|\mathbf{u}\times\mathbf{v}\|^2 &= (\mathbf{u}\times\mathbf{v})\cdot(\mathbf{u}\times\mathbf{v}) \\
&= \mathbf{u}\cdot\big(\mathbf{v}\times(\mathbf{u}\times\mathbf{v})\big) \\
&= \mathbf{u}\cdot((\mathbf{v}\cdot\mathbf{v})\mathbf{u} - (\mathbf{v}\cdot\mathbf{u})\mathbf{v}) \\
&= (\mathbf{v}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{v}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{v}) \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2 \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta)^2 \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) \\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta
\end{aligned}$$

Now take square roots to conclude that

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta.$$

(Note that $0 \le \theta \le \pi/2$, so $\sin\theta \ge 0$. Hence, we don't need absolute value signs when we take square roots.)