Math 130 Linear Algebra

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Due today. Exercises from section 5.2: 5ab, 6ab, 10ab, 13, T3.

Due Wednesday. Exercises from section 6.1: 11, 15, 20, T3.

Due Friday. Exercises from section 6.2: 1-3, 5, 6, 9, 14, 17-19.

Last time. Abstract real vector spaces with a precise definition and several examples.

For next time. Finish section 6.2 on subspaces and start 6.3: span of a set of vectors, linear independence, solution set of a homogeneous system of linear equations.

Second test. Select date.

Today. Properties of vector spaces, subspaces.

Properties of vector spaces. We defined a vector space as having only two operations, vector addition and scalar multiplication, that satisfy a certain collection of axioms. From these axioms we can prove several other properties such as the following.

a. Zero times any vector is the zero vector: $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$.

b. Any scalar times the zero vector is the zero vector: $c\mathbf{0} = \mathbf{0}$ for any real number c.

c. The only ways the product of a scalar and an vector can equal the zero vector are when either the scalar is 0 or the vector is $\mathbf{0}$.

d. For a given vector $\mathbf{v}$, there is only one vector $\mathbf{w}$ such that $\mathbf{v} + \mathbf{w} = 0$, and that vector $\mathbf{w}$ is $-\mathbf{v}$, the negation of $\mathbf{v}$.

e. The scalar -1 times a vector is the negation of the vector: $(-1)\mathbf{v} = -\mathbf{v}$.

Also, subtraction can be defined in terms of addition

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}),$$

and then all the usual properties of subtraction will follow.

Subspaces. Section 6.2 is on subspaces, that is, subsets of vector spaces that are vector spaces themselves.

Last time we looked at lots of examples of vector spaces. Some of them were subspaces of some of the others. For instance, P_n, the vector space of polynomials of degree less than or equal to n, is a subspace of the vector space P_{n+1} of polynomials of degree less than or equal to $n + 1$.

For a vector space to be a subspace of another vector space, it just has to be a subset of the other vector space, and the operations of vector addition and scalar multiplication have to be the same.

Perhaps the name "sub vector space" would be better, but the only kind of spaces we're talking about are vector spaces, so "subspace" will do.

Another characterization of subspace is the following theorem.

Theorem 1. A subset W of a vector space V is a subspace of V if and only if

(1) $\mathbf{0} \in W$;

(2) W is closed under vector addition, that is, whenever $\mathbf{w}_1$ and $\mathbf{w}_2$ belong to W, then so does $\mathbf{w}_1 + \mathbf{w}_2$ belong to W; and

(3) W is closed under scalar product, that is, whenever c is a real number and $\mathbf{w}$ belongs to W, then so does $c\mathbf{w}$ belong to W.

Yet another characterization of subspace is this theorem.

Theorem 2. A nonempty subset W of a vector space V is a subspace of V if and only if W is closed under linear combinations, that is, whenever $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ all belong to W, then so does each linear combination $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_k$ of them belong to W.

This second characterization is equivalent to the first because linear combinations are built from vector additions and scalar products, yet scalar products and vector additions are special cases of linear combinations.

Subspaces of the plane $\mathbf{R}^2$. Let's start by examining what it means to be a subspace of the vector space $\mathbf{R}^2$. This will be enough to see what the concept means.

First of all, there are a couple of obvious and uninteresting subspaces. One is the whole vector space $\mathbf{R}^2$, which is clearly a subspace of itself. A subspace is called a *proper* subspace if it's not the entire space, so $\mathbf{R}^2$ is the only subspace of $\mathbf{R}^2$ which is not a proper subspace.

The other obvious and uninteresting subspace is the smallest possible subspace of $\mathbf{R}^2$, namely the $\mathbf{0}$ vector by itself. Every vector space has to have $\mathbf{0}$, so at least that vector is needed. But that's enough. Since $\mathbf{0} + \mathbf{0} = \mathbf{0}$, it's closed under vector addition, and since $c\mathbf{0} = \mathbf{0}$, it's closed under scalar multiplication. This $\mathbf{0}$ subspace is called the *trivial* subspace since it only has one element.

So, ignoring those two obvious and uninteresting subspaces, we're left with finding all the rest, and they're the proper, nontrivial subspaces of $\mathbf{R}^2$.

And here they are. Take any line W that passes through the origin in $\mathbf{R}^2$. If you add two vectors in that line, you get another, and if multiply any vector in that line by a scalar, then the result is also in that line. Thus, every line through the origin is a subspace of the plane.

Furthermore, there aren't any other subspaces of the plane. The argument for that is fairly long. Let W be any proper, nontrivial subspace of $\mathbf{R}^2$. Since it's proper, it contains some nonzero vector $\mathbf{u}$. But W is closed under scalar multiplication, so the whole line through $\mathbf{u}$ has to be part to W. We have yet to show that if W contains another vector $\mathbf{v}$ that isn't in that line, then W has to be all of $\mathbf{R}^2$. In other words, if $\mathbf{u}$ and $\mathbf{v}$ do not lie on the same line through the origin, then the smallest subspace of $\mathbf{R}^2$ that contains them both is all of $\mathbf{R}^2$.

So, let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be two nonzero vectors that do not lie on the same line through the origin. We need to show that every other vector $\mathbf{w} = (w_1, w_2)$ is some linear combination of $\mathbf{u}$ and $\mathbf{v}$. That is to say, we need to show there exist two real numbers x and y so that $\mathbf{w} = x\mathbf{u} + y\mathbf{v}$. That means that the system of linear equations

$$\begin{aligned} u_1x + v_1y &= w_1 \\ u_2x + v_2y &= w_2 \end{aligned}$$

can be solved for x and y in terms of the other numbers. We know that such a system can be uniquely solved if the determinant of the coefficient matrix is not 0:

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \neq 0.$$

But that determinant is nonzero if and only if $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$. Thus, when $\mathbf{u}$ and $\mathbf{v}$ do not lie on the same line through the origin, then the smallest subspace of $\mathbf{R}^2$ that contains them both is all of $\mathbf{R}^2$.